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SLOVAK REPUBLIC

INTERNATIONAL CONFERENCE of SCIENTIFIC PAPER
AFASES 2012
Brasov, 24-26 May 2012

COUPLING SURFACE FLOW WITH POROUS MEDIA

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Abstract: *Simulations of flow in porous media can even help modeling tsunami interactions on a shore-line. A modeling of filtration processes would require introducing different systems of partial differential equations in the free fluid and in the porous medium regions. Such equations must be coupled through physically continuity conditions at the interface separating the two domains. We will use the well-known Beavers-Joseph interface and propose iterative methods to solve the coupling of the Navier-Stokes and Darcy equations.*

Keywords: *Navier-Stokes equations, Darcy's law, Interface conditions, iterative*

1. INTRODUCTION

A modeling of filtration processes would require introducing different systems of partial differential equations in the free fluid and in the porous medium regions. The difficulty in finding effective coupling conditions at the interface between the fluid domain and the porous layer lies in the fact that often the orders of the corresponding differential operators are different, e.g. when using Navier-Stokes and Darcy's equation. The model we consider, which is similar to the one in [1] is based on imposing the correct local equation in each region, coupled with appropriate interface conditions.

2. MODEL PROBLEM

The aim of our research is to begin the study of the following problem: an incompressible fluid in a region who can flow both ways across an interface into a domain which is a porous medium saturated with the same fluid same as [7,8, 10-15,18].

Let $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ be a bounded domain, decomposed into two non intersecting subdomains Ω_f and Ω_p separated by an interface Γ, i.e. $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\Omega_f \cap \Omega_p = \emptyset$ şi $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. We suppose the boundaries $\partial\Omega_f$ şi $\partial\Omega_p$ to be Lipschitz continuous. From the physical point of view, Γ is a surface separating the domain Ω_f filled by a fluid, from a domain Ω_p formed by a porous medium. We assume that the fluid contained in Ω_f has a fixed surface (i.e. we do not consider the free surface fluid case) and can filtrate through the adjacent porous medium.

In order to describe the motion of the fluid in Ω_p, we introduce the Navier–Stokes equations: $\forall t > 0$,

$$\partial_t u_f - \nabla \cdot T(u_f, p_f) + (u_f \cdot \nabla) u_f = f \text{ în } \Omega_f$$

$$\nabla \cdot u_f = 0 \text{ în } \Omega_f \quad (1)$$

where $T(u_f, p_f) = \nu(\nabla u_f + \nabla^T u_f) - p_f I$ is the Cauchy stress tensor, $\nu > 0$ is the kinematic viscosity of the fluid, while u_f and p_f are the fluid velocity and pressure.

∇ and $\nabla\cdot$ are, respectively, the gradient and the divergence operator with respect to the space coordinates. Moreover,

$$\nabla \cdot u = \left(\sum_{j=1}^{d} \partial_j u_{ij} \right)_{i=1,\dots,d}$$

Finally, we recall that

$$(v \cdot \nabla) w = \sum_{i=1}^{d} v_i \partial_i w$$

for all vector functions $v = (v_1, \dots, v_d)$ and $w = (w_1, \dots, w_d)$

In the domain Ω_p we define the piezometric head $\varphi = z + \frac{p_p}{\rho_f g}$ where z is the elevation from a reference level, p_p is the pressure of the fluid in Ω_p, ρ_f its density and g is the gravity acceleration.

The fluid motion in Ω_p is described by the equations:

$$\nabla \cdot u_p = 0 \text{ in } \Omega_p \qquad (2)$$

$$u_p = -K \nabla \varphi \text{ in } \Omega_p$$

where u_p is the fluid velocity, and K is the hydraulic conductivity tensor $\mathbf{K} = diag(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3)$ with $\mathbf{K} = (\mathbf{K}_{ij})_{i,j=1,\dots,d} \in L^\infty(\Omega_p)$

The first equation is Darcy's law. In the following we shall denote $K = \mathbf{K}/n$

Darcy's law provides the simplest linear relation between velocity and pressure in porous media under the physically reasonable assumption that fluid flows are usually very slowed all the inertial (nonlinear) terms may be neglected.

For the sake of clarity, in our analysis we shall adopt homogeneous boundary conditions. In particular, for the Navier–Stokes problem we impose the no-slip condition $u_f = 0$ on $\partial\Omega_f \setminus \Gamma$, while for the Darcy problem, we set the piezometric head $\varphi = 0$ on Γ_p and we require the normal velocity to be null on Γ_p $u_f \cdot \tau_j = 0$. $\mathbf{n}_p$ and $\mathbf{n}_f$ denote the unit outward normal vectors to the surfaces Ω_f and Ω_p and we have $\mathbf{n}_f = -\mathbf{n}_p$ on Γ. We suppose $\mathbf{n}_p$ and $\mathbf{n}_f$ to be regular enough. In the following we shall indicate $n = \mathbf{n}_p$ for simplicity of notation.

We supplement the Navier–Stokes and Darcy problems with the following conditions on Γ:

$$u_f \cdot n = u_p \cdot n, \qquad (3)$$

$$-n \cdot T(u_f, p_f) \cdot n = g\varphi \qquad (4)$$

$$-\tau_j \cdot T(u_f, p_f) \cdot n = \frac{\nu \alpha_{BJ}}{\sqrt{K}} (u_f - u_p) \cdot \tau_j \text{ on } \Gamma \qquad (5)$$

where $\tau_j (i = 1, \dots, d - 1)$ are linear independent unit tangential vectors to the boundary Γ, and α_{BJ} is the characteristic length of the porous medium.

Conditions (3) and (4) impose the continuity of the normal velocity on Γ, as well as that of the normal component of the normal stress, however they allow pressure to be discontinuous across the interface. The so-called Beavers–Joseph condition (5) is used here instead of Beavers–Joseph-Saffman that were mathematically proven in [17].

The coupled Navier-Stokes/Darcy model is as follows:

$$\partial_t u_f - \nabla \cdot T(u_f, p_f) + (u_f \cdot \nabla) u_f = f \; in \; \Omega_f$$

$$\nabla \cdot u_f = 0 \; in \; \Omega_f$$

$$u_p = -K \nabla \varphi \; in \; \Omega_p$$

$$\nabla \cdot u_p = 0 \text{ in } \Omega_p \qquad (6)$$

$$u_f \cdot n = u_p \cdot n \text{ pe } \Gamma$$

$$-n \cdot T(u_f, p_f) \cdot n = g\varphi \text{ on } \Gamma$$

$$-\tau_j \cdot T(u_f, p_f) \cdot n = \frac{\nu \alpha_{BJ}}{\sqrt{K}} (u_f - u_p) \cdot \tau_j \text{ on } \Gamma$$

We define the following spaces:

$H_f = \{ v \in H^1(\Omega_f)^d : \ v = 0 \ on \ \partial\Omega_f \setminus \Gamma_f \}$

$H_f^0 = \{ v \in H_f(\Omega_f) \ : \ v \cdot n = 0 \ on \ \Gamma \}$ (7)

$Q = L^2(\Omega_f), \ Q_0 = \left\{ q \in Q \ : \ \int_{\Omega_f} q = 0 \right\}$

$H_p = \{ \psi \in H^1(\Omega_p) \ : \ \psi = 0 \ on \ \Gamma_p^D \},$

$H_p^0 = \{ \psi \in H_p \ : \ \psi = 0 \ on \ \Gamma \}$

We denote by $| \cdot |_1$ şi $\| \cdot \|_1$ the H^1-seminorm and norm and by $\| \cdot \|_2$ the L^2-norm; it will always be clear form the context whether we are referring to spaces on Ω_f and Ω_p. Finally, we consider the trace space $\Lambda = H_{00}^{1/2}(\Gamma)$ şi $\| \ \|_\Lambda$ and its subspace.

Then, we introduce the bilinear forms

$$a_f(v,w) = \int_{\Omega_f} \frac{\upsilon}{2} (\nabla v + \nabla^T v) \cdot (\nabla w + \nabla^T w)$$

$$+ \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu \alpha_{BJ}}{\sqrt{K}} \left[(u + K \nabla \varphi) \cdot \tau_j \right] (v \cdot \tau_j)$$

$\forall v, w \in (H^1(\Omega_f))^d$

$$b_f(v,q) = -\int_{\Omega_f} q \nabla \cdot v \quad \forall v \in (H^1(\Omega_f))^d \quad \forall q \in Q$$

$$a_p(\varphi,\psi) = \int_{\Omega_p} \nabla \psi \cdot K \nabla \varphi \quad \forall \varphi, \psi \in H^1(\Omega_p)$$

$$a_\Gamma(v,w) = \int_\Gamma g \varphi (w \cdot n) - g \int_\Gamma \psi (v \cdot n)$$

$$a_\Omega(v,w) = a_f(v,w) + a_p(\varphi,\psi)$$

and the trilinear form

$$c_f(w;z,v) = \int_{\Omega_f} [(w \cdot \nabla) z)] \cdot v = \sum_{i,j=1}^{d} \int_{\Omega_f} w_j \frac{\partial z_i}{\partial x_j} v_i$$

$\forall v, w, y \in (H^1(\Omega_f))^d$ (8)

By integration by parts as in [20], the weak formulation for the above coupled Navier-Stokes/Darcy problem reads:

Fiind $u = (\mathbf{u}, \varphi) \in W$, $p \in Q$ *that*

$$\begin{cases} A(u,v) + C(u;u,v) + B(u,p) = F(v) \ \forall v = (\mathbf{v}, \psi) \in W \\ B(u,q) = 0 \ \forall q \in Q \quad (9) \end{cases}$$

where $A(v,w) = a_\Omega(v,w) + a_\Gamma(\varphi,\psi)$

$C(\underline{v}; w, \underline{u}) = c_f(v; w, u)$

$B(u,p) = b_f(w,q)$

$F(v) = \int_{\Omega_f} fv$

The weak formulation for the above coupled (stationary) Stokes/Darcy problem reads:

Fiind $u = (\mathbf{u}, \varphi) \in W$, $p \in Q$ *that*

$$\begin{cases} A(u,v) + B(u,p) = F(v) \ \forall v = (\mathbf{v}, \psi) \in W \\ B(u,q) = 0 \ \forall q \in Q \ (10) \end{cases}$$

Similar to [6], it is easy to verify that A(·, ·) is continuous and coercive on W and B(·,·) is continuous on $W \times Q$ and satisfies the well-known Brezzi - Babuska condition: *there exists a positive constant* $\beta > 0$ *such that* $\forall q \in Q, \ \exists w \in W$ *such that*

$b(w,q) \geq \beta \|w\|_W \|q\|_Q$

The well-posedness of the model problem (10) then follows from Brezzi's theory for saddle-point problems [4]. The continuity is obvious, while the coercivity is still a consequence of the well-known Poincare inequality and Korn inequality and using Lemma 3.2 from [9]. The bilinear functional A(·, ·) is continuous and coercive on $W \times W$ (W-elliptic) when the coefficient in the Beavers-Joseph interface boundary condition α is small enough.

3. ITERATIVE FINITE ELEMENT SOLUTION OF THE COUPLED PROBLEM

In this section, we introduce and analyze an iterative method to compute the solution of a conforming finite element approximation of (16)–(18). For the easiness of notation, we will write the algorithms in continuous form. However, they can be straightforwardly translated into a discrete setting considering conforming internal Galerkin approximations of the spaces (7).

Moreover, the convergence results that we will present hold in the discrete case without any dependence of the convergence rate on the grid parameter *h*, since they are established by using the properties of the operators in the continuous case.

We consider a triangulation T_h of the domain $\overline{\Omega_f} \cup \overline{\Omega_p}$, depending on a positive parameter $h > 0$, made up of triangles if d = 2, or tetrahedra in the three-dimensional case. We assume that the triangulations induced on the subdomains Ω_f and Ω_p are compatible on Γ, that is they share the same edges (if *d* = 2) or faces (if *d* = 3) therein.

The crucial issue concerning the finite dimensional spaces, say V_h and Q_h, approximating the spaces of velocity and pressure is that they must satisfy the discrete compatibility condition:

there exists a positive constant $\beta^* > 0$, *independent of h, such that*

$$\forall q_h \in Q_h,\ \ \exists v_h \in V_h,\ \ v_h \neq 0: \quad b_h(v_h, q_h) \geq \beta^* \|v_h\|_1 \|q_h\|_0 \qquad (30)$$

Spaces satisfying (30) are said *inf-sup stable*.

Several choices can be made in this direction featuring both discontinuous pressure finite elements (e.g., the $P_2 - P_0$ elements or the Crouzeix-Raviart elements defined using cubic bubble functions) and continuous pressure finite elements: among the latter we recall the Taylor-Hood (or $P_2 - P_1$) elements and the $P_1 iso P_2$ elements.

We have indicated by the subscript *h* the finite element approximations of u_f, p_f and φ.

The following error estimates hold. There exist two positive constants *C*1 and *C*2 such that

$E_S^h \leq C_1 h^r \left(\|u_f\|_{r+1} + \|p_f\|_r \right),\ r = 1, 2$. If $u_f \in H^{r+1}(\Omega_f)$ and $p_f \in H^r(\Omega_f)$ where $E_S^h = \|\nabla u_f - \nabla u_{fh}\|_0 + \|p_f - p_{fh}\|_0$ while $E_D^h \leq C_2 h^l \|\varphi\|_{l+1}$, $l = \min(2, s-1)$.

If $\varphi \in H^s(\Omega_p)$, $s \geq 2$ with $E_D^h = \|\varphi - \varphi_h\|_l$

3.1 Fixed – point iterations

Fixed-point iterations to solve the coupled problem (9) can be written as follows

find $u_f \in H_f, p_f \in Q, \varphi \in H_p$ such that

$$a_f(u_f^n, v) + b_f(v, p_f^n) + c_f(u_f^{n-1}, u_f^n, v) + \int_\Gamma g\varphi^n (v \cdot n)$$

$$+ \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu \alpha_{BJ}}{\sqrt{K}} \left[(u_f^n + K\nabla \varphi^n) \cdot \tau_j \right] (v \cdot \tau_j) = \int_{\Omega_f} fv$$

$$b_f(u_f^n, q) = 0$$

$$a_p(\varphi^n, \psi) = \int_\Gamma \psi (u_f^n \cdot n)$$

for all $v \in H_f, q \in Q, \psi \in H_p$

3.2 Newton-like methods

Let us consider now the Newton method to solve (the discrete form of) (9).

find $u_f^n \in H_f, p_f^n \in Q, \varphi^n \in H_p$ such that

$$a_f(u_f^n, v) + b_f(v, p_f^n) + c_f(u_f^n, u_f^{n-1}, v) + \int_\Gamma g\varphi^n (v \cdot n)$$

$$+ \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu \alpha_{BJ}}{\sqrt{K}} \left[(u_f^n + K\nabla \varphi^n) \cdot \tau_j \right] (v \cdot \tau_j)$$

$$= c_f(u_f^{n-1}, u_f^{n-1}, v) + \int_{\Omega_f} fv$$

$$b_f(u_f^n, q) = 0$$

$$a_p(\varphi^n, \psi) = \int_\Gamma \psi (u_f^n \cdot n)$$

for all $v \in H_f, q \in Q, \psi \in H_p$

We consider the computational domain $\Omega = (0,\ 1)\ \text{x}\ (0,\ 2)$ with and $\Omega_f = (0,\ 1)\ \text{x}\ (1,\ 2)$ and $\Omega_p = (0,\ 1)\ \text{x}\ (0,1)$ and uniform regular triangulations characterized by a parameter h. We use Taylor-Hood elements for the Navier-Stokes equations and quadratic Lagrangian elements for the Darcy equation .

In a first test, we set the boundary conditions in such a way that the analytical solution for the coupled problem is

$$u_f = (x^2y^2 + e^{-y}, -\frac{2}{3}xy^3 + [2-\pi\sin(\pi x)])$$

$$p_f = -[2-\pi\sin(\pi x)]\cos(2\pi y)$$

$$\varphi = [2-\pi\sin(\pi x)][-y+\cos(\pi(1-y))]$$

In order to check the behavior of the iterative methods that we have studied with respect to the grid parameter h, to start with we set the physical parameters (v, K, e, g) all equal to 1. The algorithms are stopped as soon as $\left\|x^n - x^{n-1}\right\|_{L^2} / \left\|x^n\right\|_{L^2} \le 10^{-10}$, where $\|\cdot\|_{L^2}$ is the L^2 norm and x^n is the vector of the nodal values of $\left(u_f^n, p_f^n, \varphi^n\right)$ Our initial guess is $u_f^0 = 0$.

The number of iterations obtained using the fixed-point algorithm, the Newton method are displayed in Table 1.

h	Fixed-point	Newton
2^{-2}	23	11
2^{-3}	23	11
2^{-4}	23	11
2^{-5}	23	11

All methods converge in a number of iterations which does not depend on h.
We present the CPU times. Table 2 shows the CPU times for the Navier-Stokes/Darcy model and the three methods. It is very clear that Newton algorithm is with significant reduction in computational time.

h	Fixed-point	Newton
2^{-2}	1.30×10^1	0.60×10^1
2^{-3}	1.343×10^2	1.021×10^2
2^{-4}	1.343×10^3	10.43×10^3

3. CONCLUSIONS & ACKNOWLEDGMENT

The numerical algorithms for solving the coupled system of free fluid and porous media are separated into three major categories:
- the first group of methods uses different equations in different domains, e.g., the Navier–Stokes equation in the liquid region and the Darcy model in the porous zones and couples them through suitable interface conditions. These kind of algorithms use domain decomposition techniques
- the second group consists of those algorithms, that solely uses one system of equations in the whole domain obtaining the transition between both fluid and porous regions through continuous spatial variations of properties ('single-domain approach').
- the two method grid by decoupling the mixed model by a coarse grid approximations to the interfaces conditions

It is very clear that Newton algorithm is with significant reduction in computational time. This algorithm is very good.

This work was partially supported by the strategic grant POSDRU/88/1.5/S/52826, Project ID52826 (2009), co-financed by the European Social Fund – Investing in People, within the Sectoral Operational Programme Human Resources Development 2007-2013.

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